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Properties of wavelet coefficients of self-similar time series

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Abstract — The analysis and processing of sequence of the data presented in the form of time series, is one of the prevalent methodologies in studying of various processes and the phenomena which are concerned to different fields of activity and researches. As a practice shows, the most of investigated time series have a property of self-similarity. At the same time, among existing variety of different ways and methods for the data processing which are presented in the form of a time series, it is possible to allocate the ideology of a multiresolution wavelet-analysis. The essence of such ideology consists in carrying out of wavelet-expansion of investigated data and the subsequent analysis of corresponding coefficients of such expansion – wavelets coefficients. Thus, necessity in answer to a question about properties of such wavelet coefficients is objective. It makes a basis of the given research. Hence, the basic properties of wavelet coefficients of self-similar time series are formulated and proved on theoretical level in this work. Also, there are presented the results of experiments which have confirmed a theoretical substantiation of the basic properties of wavelet coefficients of self-similar time series. They have shown that the considered properties are characteristic for time series with different correlation structure.

Index Terms — correlation structure, expansion, self-similarity, time series, wavelet coefficients, wavelet-analysis.

1 Introduction

For the solution of many practical problems representation of the initial given and received results in the form of some sequence, in particular in the form of time series is absolutely natural.

An example of such problems can be:

Research of change of dynamics of various indicators of banks functioning throughout some period of time [1];

The analysis of the data which represent results of some experiments that were carried out in strict time succession [2];

Studying of the Internet traffic bandwidth properties which are described by different characteristics of volumes of the transferred information by means of datacom speed [3];

Per line or columnwise analysis of the two-dimensional image which is presented in the form of a matrix of data about investigated object [4];

Studying and revealing of regularity among the processes connected with the different natural phenomena: floods, avalanching, spread of fires, change of volumes of water in the rivers depending on quantity of precipitations [5];

Such generalization of different data, in the form of interconnected sequence and time series allows to not only present and describe compactly of investigated data, but also to consider the objective nature of communication between separate elements of the interconnected sequence, to open the data change regularity in their analyzed sequence. Finally, it also does proved and claimed the presentation of various analyzed data in the form of their some sequence.

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At the same time, data presentation in the form of some sequence assumes use of certain procedures of their processing that also depends on nature of considered sequence and a context of a solved problem. In particular, some sequence of data could have a property of self-similarity that is quite typical for data series which describe both dynamics of various economic and technical processes, and the phenomena of natural character [6], [7], [8]. At the same time results of processing of initial sequences of the data presented in the form of time series can also have some similar properties. Such similarity of properties is caused by both the unified procedures of processing of initial data, and the similar nature of data, for example, in the form of their self-similarity. Thus considered time series of various processes and the phenomenon can have quite difficult structure, contain local features of the various form and time extent. However it does not limit possibility of consideration of the general properties of results of processing of various time series, reasoning from a generality of such property as self-similarity of considered series. It also has formed a basis for a choice of subjects, the purpose and the primary goals of the given research.

2 A Multiresolution Wavelet-Analysis as a Method of a Sequence Analysis of Investigated Data

One of the methods of the sequence analysis of the data presented in the form of time series is so-called multiresolution analysis method, on the basis of the theory of wavelet-transformations [6], [9], [10], [11], [12].

A multiresolution wavelet-analysis transforms time series to hierarchical structure by means of the wavelet-transformations which results to the set of wavelet-coefficients. On each new level of wavelet-expansion there is a division of an approximating signal of the previous level of detail (presented by some time series) on its high-frequency component and on more smoothed approximating signal [10, 12]. The number of readout in investigated process, and, hence, the number of coefficients decreases every time in 2 times at increase of detail level on a unit.

Thus, a multiresolution analysis consists in splitting of an investigated number into two components – approximating and detailing, with their subsequent crushing for the purpose of change of level of expansion of a signal to the set level of expansion. The number of practically used wavelets on j scale coefficient sets level of expansion of a signal [10]. In practice, it is quite spread the use of discrete wavelet transformation (DWT) [4]. It is connected with that application DWT becomes especially effective when the signal has high-frequency components of short duration and extensive low-frequency components. Such signals are also met more often in practice [10], [12].

Discrete wavelets are used, as a rule, in steam with the discrete scaling-functions $\varphi_{j,k}(t)$ [10], [12], [13], [14]. Scaling-functions have the general area of the value assignment with wavelets and define ratio between such values (form). According to discrete wavelet-transformation time series $X(t)$, ($t = t_1, t_2, ...$) consists of a set of coefficients – detailing and approximating [13], [14]:

$$X(t) = \sum_{k=1}^{N_a} apr(N,k)\varphi_{N,k}(t) + \sum_{j=1}^{N}\sum_{k=1}^{N_j} det(j,k)\psi_{j,k}(t), \quad (1)$$

where :

$apr(N,k)$ – Approximating wavelet-coefficients of level N;

$det(j,k)$ – Detailing wavelet-coefficients of level j;

N – Chosen maximum level of expansion;

N_j – Quantity of detailing coefficients at j level of expansion;

N_a – Quantity of approximating coefficients at level N;

$\psi(t)$ – Mother wavelet-function;

$\varphi(t)$ – Corresponding scaling-function.

At set the mother wavelet and $\psi(t)$ corresponding scaling-function approximating $\varphi(t)$ coefficients and $a_X(j,k)$ detailing coefficients of $d_X(j,k)$ DWT for the process $X(t)$ can be defined as follows [13], [14]:

$$a_X(j,k) = \int_{-\infty}^{+\infty} X(t)\varphi_{j.k}(t)dt, \quad (2)$$

$$d_X(j,k) = \int_{-\infty}^{+\infty} X(t)\psi_{j.k}(t)dt, \quad (3)$$

In particular, on each level of discrete wavelet transformation (DWT) detailing coefficients represent features, detail of the investigated signal, arising at transition from one scale to another and are equal [9], [15]:

$$d_X(j,k) = \langle X, \psi_{j,k} \rangle, \quad (4)$$

where:

$d_X(j,k)$ – Detailing wavelet-coefficients $k = \overline{1, N_j}$ on level j,

$\langle X, \psi_{j,k} \rangle$ – Scalar product of investigated sequence of data in the form of time series $X(t)$ and a mother wavelet ψ on corresponding level of expansion j.

Thus, the main tool for the research of studied processes is processing of the wavelet- coefficients which have been received on different scales. Besides, allocated factors allow to localize places of heterogeneity and differences of analyzed signals, that is to carry out its spatial division on areas with prominent features. As a result of DWT received series of coefficients has the define properties which allow to investigate behaviour of the stochastic processes that have the properties of self-similarity. These properties are the direct subject of the given research. However before we consider them, let's make some additional remarks which will be useful during the subsequent statement of a material.

The characteristic of a measure of self-similarity of stochastic process at the multiresolution wavelet-analysis are the values of Hurst's indicator which is an indicator of complexity of dynamics and correlation structure of time series [9], [15].

At the same time stochastic process $X(t)$ with a continuous variable of time is called as self-similar in a narrow sense with parameter H, $0 < H < 1$ if for any material value $a > 0$ finite-dimensional distribution for $X(at)$ are identical for finite-dimensional distributions of $a^{-H}X(at)$ [9], [16].

In other words, if for any $k \geq 1$, $t_1, t_2, ..., t_k$ and any $a > 0$ [9, 16, 17]:

$$(X(t_1), X(t_2), ...X(t_k),) \cong$$
$$\cong (a^{-H}X(at_1), a^{-H}X(at_2), ..., a^{-H}X(at_k))' \quad (5)$$

or

$$\{X(t), t \in \{t_1, t_2, ..., t_k\}\} \cong \{a^{-H}X(at), t \in \{t_1, t_2, ..., t_k\}\}, \quad (6)$$

where:

$\cong$ – Means equality of finite-dimensional principles of distribution.

H – Hurst's indicator which represents a measure of self-similarity of stochastic process.

In particular the formula (6) shows, that change of time scale is equivalent to change of spatial scale of statuses. Therefore, typical realizations of self-similar process are visually similar irrespective of a time scale on which they are considered. However it doesn't mean, that the process is repeated in accuracy, it is more likely observed a similarity of statistical properties, because of statistical characteristics do not vary at scaling [9], [16].

Stochastic process $X(t)$ is statistically self-similar or strictly self-similar in a broad sense, if the process $a^{-H}X(at)$ has the same statistical characteristics of the second order (expectation value, dispersion and autocorrelation function), as $X(t)$ [9], [16]:

$$M[X(t)] = a^{-H} \cdot M[X(at)], \quad (7)$$

$$D[X(t)] = a^{-2H} \cdot D[X(at)], \quad (8)$$

$$r_x(t,s) = a^{-2H} \cdot r_x(at, as), \quad (9)$$

where:

$M[X(t)]$ – expectation value of the process $X(t)$,

$D[X(t)]$ –dispersion of the process $X(t)$,

$r_X(t,s)$ – autocorrelation function (correlation function),

t, s – time moments.

One of qualitative and quantitative characteristics of investigated sequence of data in the form of time series is the spectrum of its wavelet-energy [17].

The wavelet-energy characterizes energy of a signal on each of expansion levels that corresponds to a certain range of frequencies. Value of wavelet-energy of corresponding level shows the powerful of range of frequencies in a signal. At the same time the wavelet-energy spectrum visually displays frequency structure of a signal.

The energy value E_j on the set level of wavelet-expansion j with quantity of detailing wavelet-coefficients N_j makes [16], [17]:

$$E_j = \frac{1}{N_j}\sum_{k=1}^{N_j}(d_X(j,k))^2. \qquad (10)$$

The above listed explanations and designations allow to formulate the basic properties of wavelet-coefficients of self-similar time series.

3 Properties of wavelet-coefficients of self-similar time series

Property 1.

If casual process $X(t)$ is self-similar process with stationary increments (SPSI), then detailing coefficients $d_X(j,k)$, $k = \overline{1, N_j}$ on each level of expansion j are self-similar. It means equality of principles of distribution for series of wavelet-coefficients on each level of expansion with some scale:

$$\begin{aligned}&(d_X(j,0), d_X(j,1),\ldots,d_X(j,N_j-1)) \cong \\ &\cong 2^{j(H+\frac{1}{2})}(d_X(0,0), d_X(0,1),\ldots,d_X(0,N_j-1))\end{aligned}. \qquad (11)$$

This property of detailing coefficients follows from self-similitude of the process defined by property of scaling (compression/stretching) of mother wavelets. So if some SPSI is self-similar – $X(2^j u) \cong 2^{jH}X(u)$ then [16, 17]:

$$\begin{aligned}d_X(j,k) &= \int X(u)\psi(2^j u - k)2^{\frac{j}{2}}du = \\ &= \int 2^{\frac{j}{2}}X(2^j u)\psi(u-k)du \cong \\ &\cong 2^{j(H+\frac{1}{2})}\int X(u)\psi(u-k)du = \\ &= 2^{j(H+\frac{1}{2})}d_X(0,k)\end{aligned}. \qquad (12)$$

Property 2.

The wavelet-coefficients received as a result of expansion of process with stationary increments are stationary on each scale 2^j.

This property follows from property of wavelet-functions $\int\psi(t)dt = 0$ [16], [17] and it guarantees stationarity of coefficients for SPSI.

For stationary detailing coefficients DWS for the process with stationary p-increments, it is necessary to have zero moments of wavelet-functions of p order.

At the same time the wavelet-function which has p-zero moments allows to analyze more thin high-frequency structure of a signal by suppressing of slowly changing components of a signal (polynomial trend of $p-1$ order) [9], [17]. Considering, that in practice there are processes with stationary increments of p order, it is necessary to choose the mother wavelet $\psi(t)$ which have $n_\psi > p$ zero moments then the detailing coefficients $d_X(j,k)$ received at expansion, will be stationary on each level of expansion.

Property 3.

If there are moments of p order then for the wavelet-coefficients which were received as a result of expansion of process $X(t)$, the following equality is carried out:

$$M\left[|d_X(j,k)|^p\right] = M\left[|d_X(0,k)|^p\right]2^{jp(H+\frac{1}{2})}. \qquad (13)$$

The conclusion of the formula (13) is based on the formula (12) and properties of expectation value:

$$\begin{aligned}M\left[|d_X(j,k)|^p\right] &= M\left[|d_X(0,k)|^p 2^{j(H+\frac{1}{2})}\right] = \\ &= M\left[|d_X(0,k)|^p\right]M\left[2^{j(H+\frac{1}{2})}\right] = \\ &= M\left[|d_X(0,k)|^p\right]2^{j(H+\frac{1}{2})}\end{aligned}. \qquad (14)$$

As consequence, for the processes with a final dispersion the expression (13) will be transformed into the following form:

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$$M\left[\left|d_X(j,k)\right|^2\right] = M\left[\left|d_X(0,k)\right|^2\right]2^{j(2H+1)}. \quad (15)$$

Property 4.

If $X(t)$ is SPSI, the correlation function of wavelet-coefficients of j level decreases according with a ratio:

$$M[d_X(j,k)d_X(j,k+n)] \cong n^{2(H-n_\psi)},\ n \to \infty, \quad (16)$$

then the higher number of zero moments n_ψ of a mother wavelet ψ, the faster aspires to zero the correlation function r_X.

Property 5.

For the different levels of expansion $j_1 \neq j_2$ and for all n correlation of detailing coefficients of these levels $d_X(j_1,k)$ and $d_X(j_2,k+n)$ is equal to 0.

Property 6.

Detailing coefficients of DWS on each level of expansion j have normal distribution with a zero average $N(0,\sigma)$.

4 Results of Experimental Researches

The above listed properties of wavelet-expansion can be used for the analysis of various real time series in practice.

In particular there was carried out the comparative analysis of wavelet-expansion of series where stochastic components have a different correlation structure: signal-independent noise, processes of autoregress with short-term dependence, fractal processes with long-term memory and non-stationary fractal processes with a trend component. Researches were spent for various modelling signals with small length of sample of an investigated series ($L = 128, 256, 512$).

The process with short-term dependence was modeled by the process of autoregress of the first order [9]:

$$X(t) = \phi_1 X(t-1) + \varepsilon(t), \quad (17)$$

where :

$-1 < \phi_1 < 1$ – Some constant;

t – Discrete time;

$\varepsilon(t)$ – Independent from $X(t)$ values of a normal random variable $N(0,\sigma)$.

Gaussian fractal noise with Hurst's indicator H=0.8 has been used as a model of fractal noise.

Non-stationary fractal process was modeled by a signal which represents the sum of fractal noise and a polynom of a different degree.

Signal-independent noise is presented by classical Brown movement with Hurst's indicator H= 0.5.

The wavelet Dobeshi of 4th order – db4 was used for the research of processes with a different correlation structure.

In figures 1-4 modelling time series (a), corresponding functions of correlation (b) and a wavelet-energy spectrum (c) are presented.

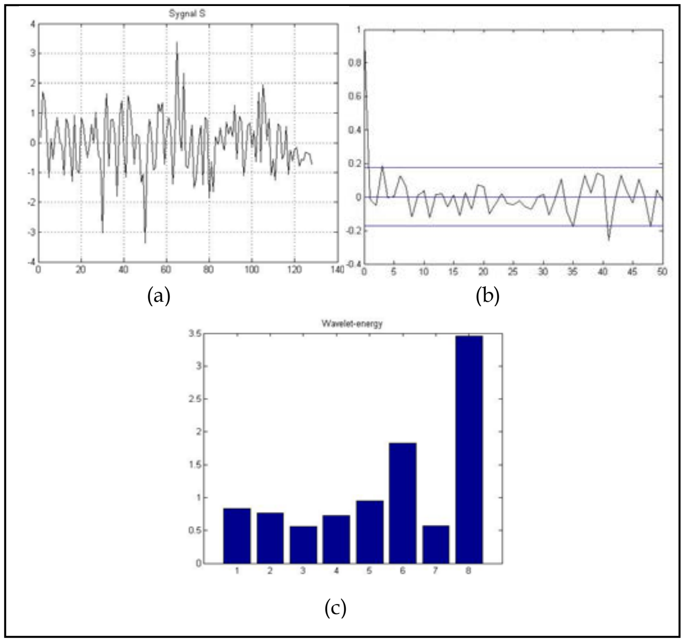

Fig. 1: A modelling time series – signal-independent noise

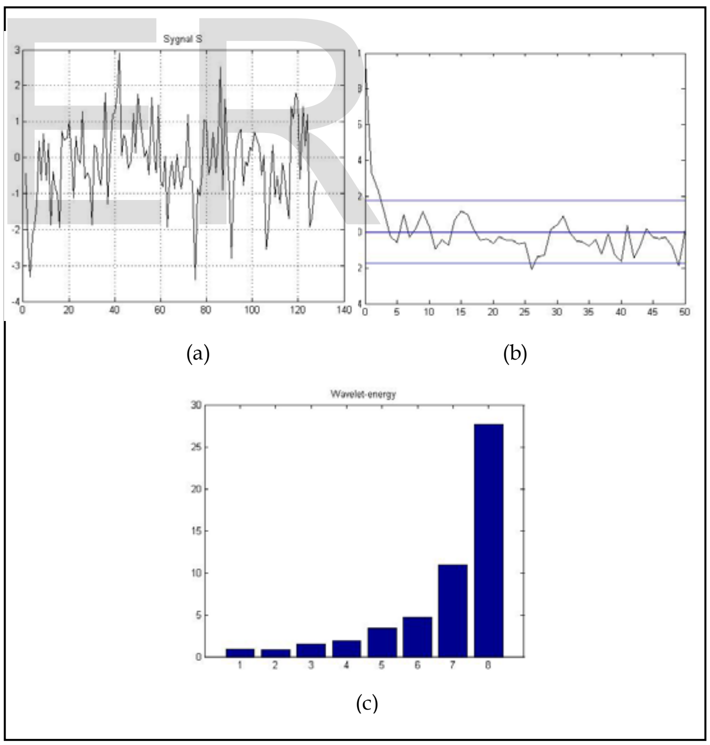

Fig. 2: A modelling time series – autoregress process

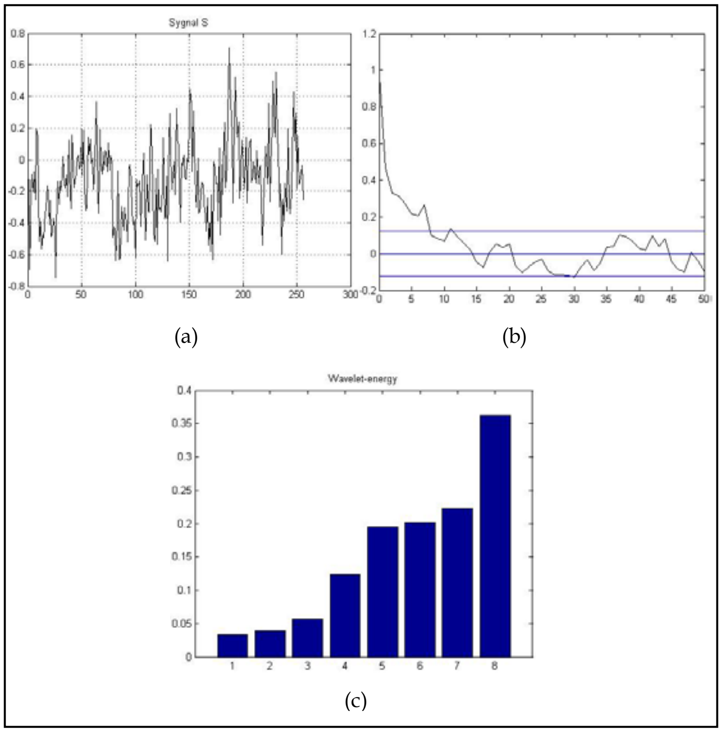

Fig. 3: A modelling time series – fractal noise

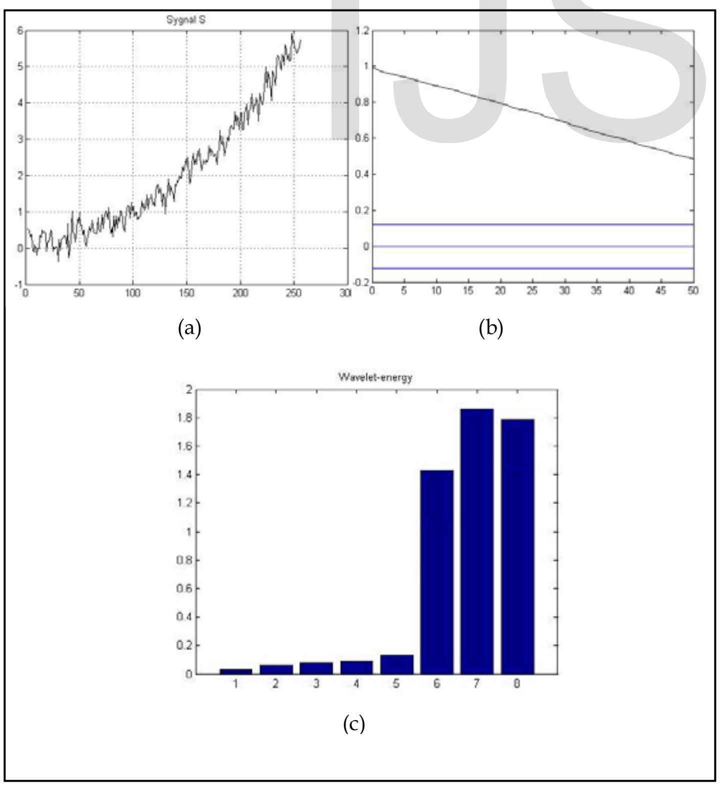

Fig. 4: A modelling time series – the non-stationary process which is presented by a polynomial trend with noise

The confidential intervals of a significance value in 95 % are shown with horizontal lines on the schedules of Figure 1 to Figure 4 of the correlation functions (b). For all modelling signals, except a signal with a trend, autocorrelation coefficients lay in corresponding limits. The number of readout 128 and 256 has been chosen for modeling signals with Hurst's theoretically set indicator, H =0.75.

Polynoms with different degree have been also generated as a trend in the investigated series.

For the casual process of autoregress correlation function exponentially quickly decreases to zero. Such dependence is characteristic for many casual processes with short-term memory.

For the fractal noise, with Hurst's indicator equal 0.75 the correlation loss goes under the hyperbolic law that allows to do a conclusion about presence of "long memory" in its dynamics.

As it specified above in Figure1 to Figure 4 the corresponding spectra of wavelet-energy which were received with expansion of modelling signals by a mother wavelet db4 at 8 levels of expansion are also displayed. From presented above it is possible to see, that wavelet-spectrum of fractal and signal-independent noise have frequencies of all range equally (Fig. 1 and Fig. 3). For the processes of autoregress where the slow fluctuations which are defined with the large coefficient of autoregress prevail, the spectrum shows the big splash on low frequencies and absolutely insignificant on high frequencies (Fig. 2).

If there is a trend in investigated signal (Fig. 4) then on low frequencies the trend gives the big splash in coefficients. Thus, influence of low-frequency components is appreciable.

During research properties of detailing and approximating coefficients have been also analyzed at various levels of the wavelet expansion and depicted in the figures such as (Figure 5 to Figure 8), where; (a) is for the function of autocorrelation of approximating coefficients of expansion and (b) is for the function of autocorrelation of detailing coefficients of expansion). Investigated modeling time series correspond to time series which were presented on Figure 1 to Figure 4 found in good agreement.

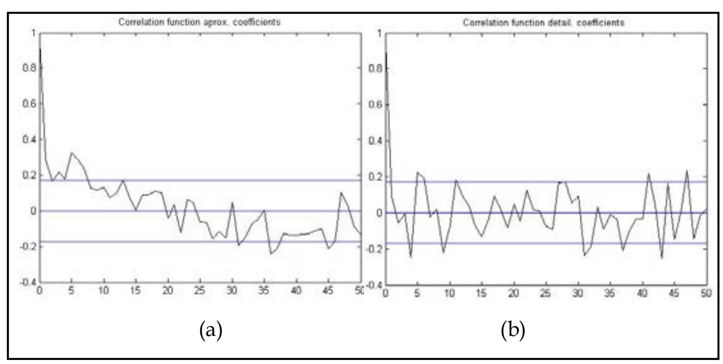

Fig. 5: Functions of autocorrelation for approximating and detailing coefficients of wavelet-expansion of modeling time series – signal-independent noise

Further the correlation dependence of wavelet coefficients have been calculated mathematically and it shows that non-stationary properties are traced in a set of approximating coefficients. It is clearly mentioned in Figure 5 to Figure 8.

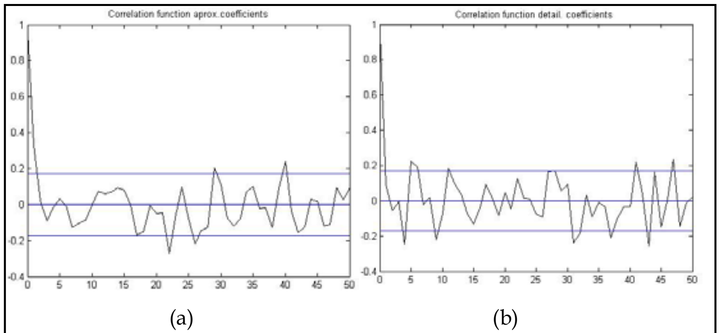

Fig. 6: Functions of autocorrelation for approximating and detailing coefficients of wavelet-expansion of a modeling time series – autoregress process

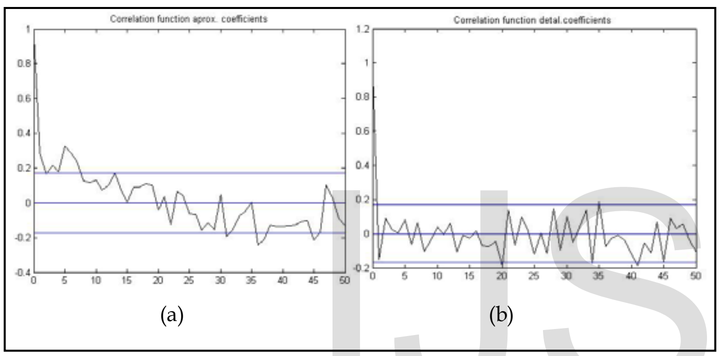

Fig. 7: Functions of autocorrelation for approximating and detailing coefficients of wavelet-expansion of a modeling time series – fractal noise

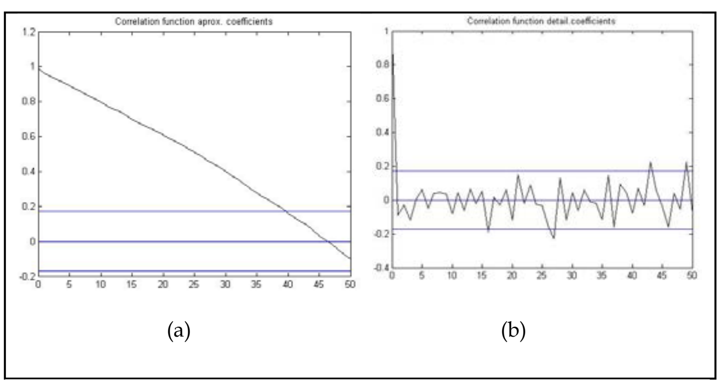

Fig. 8: Functions of autocorrelation for approximating and detailing coefficients of wavelet-expansion of modeling time series – the non-stationary process which is presented by a polynomial trend with a noise

The autocorrelation function of approximating coefficients becomes a slowly decreasing curve (as it is in Fig. 8) for a polynomial trend with a noise.

Autocorrelation function of fractal noise shows slow fading under the hyperbolic law (Fig. 7), for the processes with short-term memory autocorrelation function quickly decreases to 0 and shows a decrease according to exponential law (Figure 5 and Figure 6).

Both laws of distribution for detailing, and approximating coefficients at different levels of expansion have been numerically investigated. It has been found out, that, detailing coefficients of DWS on each level of expansion j have normal distribution with a zero average.

At the same time researches have shown that only detailing coefficients are normally distributed. Practically for all selective data the hypothesis has been accepted with a significance value, $\alpha = 0.05$.

Finally, in we have drawn some histograms of distribution of approximating and detailing coefficients for 8 level expansions for a mother wavelet – db4. And we have depiceted in Figures 9 to12, in which (a) is the histograms of distribution approximating and (b) is the histograms of distribution of detailing expansion) in all Figures 9-12.

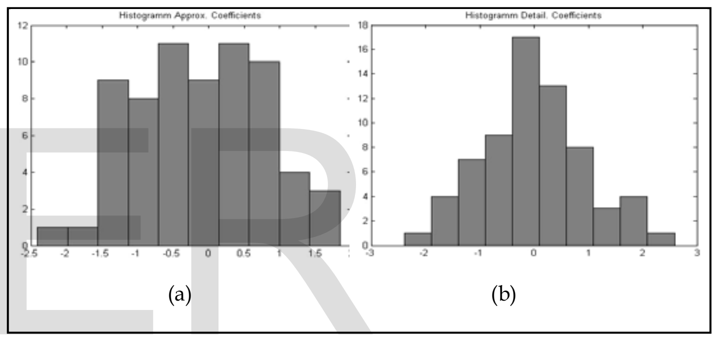

Fig. 9: Histograms of distribution of approximating and detailing coefficients for investigated time series – signal-independent noise

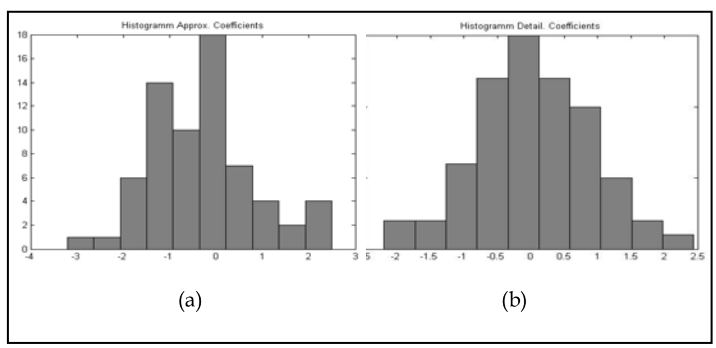

Fig. 10: Histograms of distribution of approximating and detailing coefficients for investigated time series – autoregress process

Apparently from presented above, results of experiments completely confirm theoretical calculations concerning properties of wavelet- coefficients of self-similar time series. At the same

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time it is possible to see that such properties are characteristic for time series with a different correlation structure.

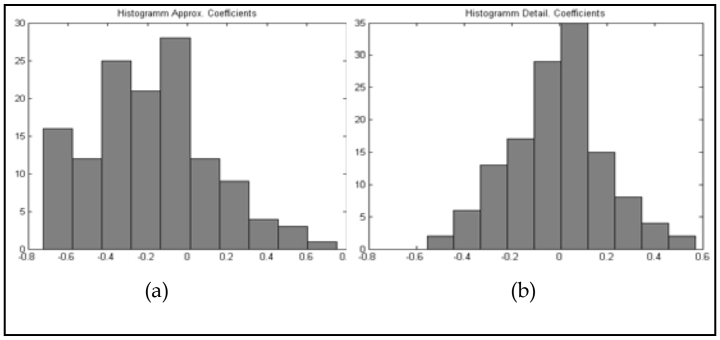

Fig. 11: Histograms of distribution of approximating and detailing coefficients for investigated time series – fractal noise

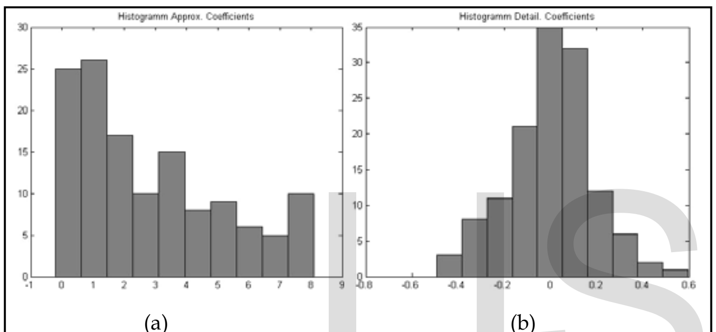

Fig. 12: Histograms of distribution of approximating and detailing coefficients for investigated time series – the non-stationary process which is presented by a polynom with a noise

5 Conclusions

First of all, the basic moments, concerning use of a multi-resolution wavelet-analysis as a method of the sequence analysis of the investigated data presented in the form of self-similar time series are considered in work. Such consideration has caused necessity of carrying out of research of properties of wavelet-coefficients of self-similar time series.

At theoretical level the basic properties of wavelet-coefficients of self-similar time series are formulated and proved.

These experiments have allowed to confirm a theoretical substantiation of the basic properties of wavelet-coefficients of self-similar time series. In particular, properties of approximating and detailing wavelet-coefficients, which had been derived on different scales of discrete wavelet-expansion, have been investigated. At that, it has been shown that theoretically proved properties of wavelet-coefficients of self-similar time series are characteristic for time series with a different correlation structure. It allows to use the considered properties of wavelet-coefficients of self-similar time series in researches for defferent applied problems.

References

[1] G. P. Kots and V. V. Lyashenko, "Banking sectors of the economies of European countries in the representation of statistical interrelation between indices that characterize their development". *European Applied Sciences*. Vol. 1, pp. 461-465 (2012).

[2] A. Khan, S. Joshi, M. Ahmad and V. Lyashenko, "Some Effect of Chemical Treatment by Ferric Nitrate Salts on the Structure and Morphology of Coir Fibre Composites". *Advances in Materials Physics and Chemistry*. Vol. 5, pp. 39-45 (2015).

[3] A. Karpukhin, I. Kudryavtsev, A. Borisov, D. Gritsiv and H. Cho, "Computer Simulation of Chaotic Phenomena in High-Speed Communication Networks". *Journal of Korean Institute of Information Technology*. Vol. 11(3), pp. 113-122, (2013).

[4] V. Lyashenko, O. Kobylin and M. A. Ahmad, "General Methodology for Implementation of Image Normalization Procedure Using its Wavelet Transform". *International Journal of Science and Research (IJSR)*. Vol. 3(11), pp. 2870-2877 (2014).

[5] M. Keblouti, L. Ouerdachi and H. Boutaghane, "Spatial interpolation of annual precipitation in Annaba-Algeria-comparison and evaluation of methods". *Energy Procedia*. Vol. 18, pp. 468-475 (2012).

[6] L. Delbeke and P. Abry, "Stochastic integral representation and properties of the wavelet coefficients of linear fractional stable motion". *Stochastic Processes and their Application*. Vol. 86 (2), pp. 177–182 (2008).

[7] W. E. Leland, M. S. Taqqu, W. Willinger and D. V. Wilson, "On the self-similar nature of Ethernet traffic (extended version)". *Networking, IEEE/ACM Transactions on*. Vol. 2(1), pp. 1-15 (1994).

[8] M. W. Garrett and W. Willinger, "Analysis, modeling and generation of self-similar VBR video traffic". *ACM SIGCOMM Computer Communication Review*. Vol. 24 (4), pp. 269-280 (1994).

[9] L. Kirichenko, T. Radivilova and Zh. Deineko, "Comparative Analysis for Estimating of the Hurst Exponent for Stationary and Nonstationary Time Series". *Information Technologies & Knowledge*. Vol. 5 (1), pp. 371–388 (2011).

[10] S. Mallat and W. L. Hwang, "Singularity detection and processing with wavelets". *IEEE Transactions on Information Theory*. Vol. 38 (2), pp. 617–643 (1992).

[11] P. Abry, R. Baraniuk and P. Flandrin, "The multiscale nature of network traffic: discovery analysis and modeling". *IEEE Signal Processing Magazine*. Vol 4 (2), pp. 5–18 (2002).

[12] P. Flandrin, "Wavelet analysis and synthesis of fractional Brownian motion". *IEEE Transactions on Information Theory*. Vol. 38, pp. 910–917 (1992).

[13] N. Kingsbury, "Image processing with complex wavelets". *Philosophical Transactions of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences*. Vol. 357 (1760), pp. 2543-2560 (1999).

[14] C. E. Heil and D. F. Walnut, "Continuous and discrete wavelet transforms". *SIAM review*. Vol. 31(4), pp. 628-666 (1989).

[15] L. O. Kirichenko, Zh. V. Deineko and T. A. Radivilova, "Application of DFA Method in Fractal Analysis of Time Series of Different Nature". *Modern Stochastics: Theory and Applications*. International Conference, pp. 9–10 (2010).

[16] P. Abry and F. Sellan, "The wavelet-based synthesis for the fractional Brownian motion proposed by F. Sellan and Y. Meyer: Remarks and fast implementation". *Applied and Computational Harmonic Analysis*. Vol 3 (4), pp. 377–383 (1996).

[17] B. B. Mandelbrot and J. R. Wallis, "Robustness of the Rescaled Range R/S in the Measurement of Noncyclic Long-Run Statistical Dependence". *Water Resources Research*. Vol. 5, pp. 967–988 (1969).